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PERTURBATION METHOD
IN THE THEORY OF
NONLINEAR OSCILLATIONS

by Ahmed Aly Kamel

Prepared by
STANFORD UNIVERSITY
Stanford, Calif.
for



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THEORY OF NONLINEAR OSCILLATIONS

By ^{Enit}Ahmed Aly Kamel

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ABSTRACT

Perturbation expansions for treating nonlinear oscillation problems are presented. These expansions are based on a transformation similar to that of the Lie transforms described by Deprit. A combination of the obtained expansions and expansions based on Lie transforms is also suggested for those problems which are mainly represented by a Hamiltonian with some smaller perturbing forces nonderivable from a potential.

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ILLUSTRATION

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1. Recursive transformation of an analytic function
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Chapter I

INTRODUCTION

Consider the nonlinear oscillatory dynamical system represented by the Lagrange equations

$$\frac{d}{dt} \mathcal{F}_{\dot{q}} - \mathcal{F}_q = \epsilon Q'(q, \dot{q}, t; \epsilon) \quad , \quad (1)$$

where q is the generalized coordinate vector, t is the independent variable, ϵ is a small parameter, $\mathcal{F}(q, \dot{q}, t; \epsilon) = T - V$ is the Lagrangian, T and V are the system kinetic and potential energies, and $\epsilon Q'(q, \dot{q}, t; \epsilon)$ is the generalized perturbing force vector nonderivable from a potential.

Now, define the generalized momentum vector as

$$p \equiv \mathcal{F}_{\dot{q}} \quad , \quad (2)$$

and the Hamiltonian R as

$$R \equiv p \cdot \dot{q} - \mathcal{F} \quad . \quad (3)$$

In view of (1) and (2), the variation δR of R has the form

$$\delta R = - \left[\dot{p} - \epsilon Q(q, p, t; \epsilon) \right] \cdot \delta q + \dot{q} \cdot \delta p \quad , \quad (4)$$

where $Q(q, p, t; \epsilon)$ is the vector formed by substituting the $\dot{q}(q, p, t)$ obtained from (2) into $Q'(q, \dot{q}, t; \epsilon)$. Equation (4) now yields

$$\dot{q} = R_p \quad (5a)$$

$$\dot{p} = - R_q + \epsilon Q(q, p, t; \epsilon) \quad . \quad (5b)$$

Many of the problems which can be formulated mathematically like (5) cannot be solved exactly for an arbitrary ϵ , hence their solutions might be sought by perturbation methods. Here, as a powerful perturbation method, we consider the method of variation of parameters. The first step in this method is to find $\epsilon = \epsilon_0$ (ϵ can be normalized so that $\epsilon_0 = 0$) for which the reduced problem

$$\dot{q} = R_p(q, p; 0) \quad (5c)$$

$$\dot{p} = -R_q(q, p; 0) \quad (5d)$$

can be solved exactly in the canonical form

$$q = q(x, X) \quad (6)$$

$$p = p(x, X) .$$

Here, x and X are the new generalized coordinate and momentum vectors, respectively, related to the constants of motion of the unperturbed system.

The second step in the method of variation of parameters is to consider (6) a canonical transformation for the original problem (5). Since this transformation is canonical, it should satisfy the constraint

$$\dot{q} \cdot \delta p - \dot{p} \cdot \delta q - \delta R = \dot{x} \cdot \delta X - \dot{X} \cdot \delta x - \delta H , \quad (7)$$

where

$$H = R \Big|_{\substack{q=q(x, X) \\ p=p(x, X)}}$$

is the new Hamiltonian. Now, to allow for the additional vector $\epsilon Q(q, p, t; \epsilon)$ in (5b), we should add $\epsilon Q \cdot \delta q$ to both sides of (7) which now vanishes due to (4); this leads to the standard form

$$\dot{x} = H_X - \epsilon Q \cdot q_X, \quad (8)$$

$$\dot{X} = -H_X + \epsilon Q \cdot q_X,$$

where

$$Q = Q(q, p, t; \epsilon) \Big|_{\substack{q=q(x, X) \\ p=p(x, X)}} \quad (9)$$

If we define the functions f_1 and f_2 such that

$$f_{1X} \equiv -Q \cdot q_X \quad (10)$$

$$f_2 \equiv Q \cdot q_X + f_{1x},$$

then an equivalent way of writing (8) is

$$\begin{aligned} \dot{x} &= (H + \epsilon f_1)_X \\ \dot{X} &= -(H + \epsilon f_1)_x + \epsilon f_2. \end{aligned} \quad (11)$$

For (8) or (11), the third step in the method of variation of parameters is to transform to new coordinates and momenta in order to eliminate some undesirable terms (e.g., short-period terms) from the right-hand side of these equations. This elimination reduces (8) or (11) to a simpler form which, in most cases, can be solved to conclude the fourth step in the method of variation of parameters.

For the case when $f_2 = 0$ (or $Q = 0$), the theory of perturbation, based on Lie transforms suggested by Deprit [1], can be used [2]. On the other hand, if $f_2 \neq 0$ (or $Q \neq 0$), the above theory is no longer applicable due to the fact that more than one independently chosen generating function is needed in order to eliminate all the undesirable terms from (8) or (11).

In some problems, noncanonical variables represent a more convenient choice over the canonical ones [3,4]; consequently, one should use a non-Hamiltonian formulation and apply a method of averaging like that of Krylov and Bogoliubov [5] to eliminate the undesirable terms (e.g., short-period terms).

When the Krylov-Bogoliubov method is applied to the majority of the problems of theoretical physics, there is no need for the computation of the effect of higher orders. Normally, only the effects of the first and second orders, rarely those of the third order, are computed. The standard representation of the method does not go beyond these limits. However, in some problems where the formal solution is slowly converging, this accuracy is insufficient and higher-order approximations are needed to secure the necessary accuracy. A computation scheme for these higher-order effects for certain nonlinear resonant problems was first obtained by Musen [6].

Recently, it was discovered that the algebraic analysis can be carried out on the computer [7,8]. Consequently, simplified formulae suitable for this purpose, as well as reductions in the computation requirements, are now desired. Further exposition is aimed toward these achievements. In generating our perturbation scheme, we shall use a non-Hamiltonian formulation to apply the resulting algorithm to a wider class of problems.

In the present method, we differ from Krylov-Bogoliubov [5] and Musen [6] in the way we obtain the perturbation expansions. Here, the expansions are based on a transformation generated by a vector W similar to the generating function in canonical transformations depending on a small parameter [1,2]. This technique has the advantage of obtaining simplified general expansions for the construction of the transformed differential equations, the forward and the inverse mappings, and any function of the original variables in terms of the new variables.

Due to the similarity between the proposed transformation and the Lie transforms [1], it is shown that the theory of perturbation, based on Lie transforms, is an important, special case of the present theory.

In this special case, the generating vector W is derived from a scalar generating function so that the canonic form of the differential system of equations is preserved throughout the transformation. In the concluding sections of this exposition, the procedure is outlined and two examples are presented.

Chapter II

GENERAL EXPANSIONS

It is assumed that the first and second steps of the method of variation of parameters, outlined in the introduction (see also the example given in Chapter V), led to the system of differential equations in the standard form

$$\dot{x} = f(x; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} f_n(x) \quad . \quad (12)$$

In the above equation, x stands for a state vector which represents the constants of motion of the undisturbed system and possibly the independent variable t .

In the third step of the method of variation of parameters, it is desirable to transform the original state vector x to a new vector \bar{x} which satisfies a simpler system of differential equations, i.e.,

$$\dot{\bar{x}} = g(\bar{x}; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} g_n(\bar{x}) \quad , \quad (13)$$

in which $g(\bar{x}; \epsilon)$ contains only some desirable terms. As will become clearer in the course of the analysis, such a desirable transformation can be generated by using the generating vector $W(x; \eta)$, defined by the differential equations

$$\frac{dx}{d\eta} = W(x; \eta) \quad , \quad (14)$$

whose initial conditions at $\eta = 0$ are $x = \bar{x}(t; \epsilon)$. Note that η is a varying small parameter and we seek solutions at $\eta = \epsilon$.

Now, take any indefinitely differentiable vector $F(x; \epsilon)$ that can be expressed in terms of x and ϵ as a power series in ϵ , in the form

$$F(x; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F_n(x) \quad , \quad (15a)$$

where

$$F_n(x) = \left[\frac{\partial^n}{\partial \eta^n} F(x; \eta) \right]_{\eta=0} \quad . \quad (15b)$$

Then, in terms of \bar{x} and ϵ as a power series in ϵ , this takes the form of

$$F(x; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} F^{(n)}(\bar{x}) \quad , \quad (16a)$$

where

$$F^{(n)}(\bar{x}) = \left[\frac{d^n}{d\eta^n} F(x; \eta) \right]_{\eta=0} \quad , \quad (16b)$$

and

$$\frac{dF}{d\eta}(x; \eta) = \frac{\partial F}{\partial \eta} + F_x \cdot \frac{dx}{d\eta} \quad . \quad (17)$$

Note that $F_0(x) = F(x; 0)$ and $F^{(0)}(\bar{x}) = F(\bar{x}; 0)$. Now, given the sequence of vectors $F_n(x)$ of (15a), we wish to construct the corresponding sequence of vectors $F^{(n)}(\bar{x})$ of (16a). This will be the subject of what follows.

Using (14), equation (17) can be written as

$$\frac{dF}{d\eta}(x; \eta) = \frac{\partial F}{\partial \eta} + L_W F, \quad (18)$$

where L_W is a linear operator defined by

$$L_W F(x; \eta) = F_x \cdot W(x; \eta). \quad (19)$$

In particular, for the generating vector W of the form

$$W(x; \eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} W_{n+1}(x) \quad (20)$$

and for $F(x; \epsilon)$ of the form given by (15a), Eq. (18) yields

$$\frac{dF}{d\eta}(x; \eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} F_n^{(1)}(x), \quad (21)$$

where

$$F_n^{(1)}(x) = F_{n+1} + \sum_{m=0}^n C_m^n L_{m+1} F_{n-m}, \quad (22a)$$

$$C_m^n = \frac{n!}{m!(n-m)!}, \quad (22b)$$

and

$$L_i F(x) = F_x \cdot W_i(x), \quad i \geq 1. \quad (22c)$$

In general, for $k \geq 1$ and $n \geq 0$, one obtains

$$\frac{d^k}{d\eta^k} F(x; \eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!} F_n^{(k)}(x), \quad (23)$$

where

$$F_n^{(k)}(x) = F_{n+1}^{(k-1)} + \sum_{m=0}^n C_m^n L_{m+1} F_{n-m}^{(k-1)} . \quad (24)$$

By allowing $\eta = 0$ in the above equation, the following recursion formula is obtained; except for the redefinition of the operator L , this formula is essentially the same as Deprit's equation [2].

$$F_n^{(k)}(\bar{x}) = F_{n+1}^{(k-1)} + \sum_{m=0}^n C_m^n L_{m+1} F_{n-m}^{(k-1)} , \quad (25)$$

where

$$L_i F(\bar{x}) = F_{\bar{x}} \cdot W_i(\bar{x}) , \quad i \geq 1 . \quad (26)$$

In the above recursion relation, we have

$$F_n^{(0)}(\bar{x}) = F_n(\bar{x}) , \quad \text{and} \quad F_0^{(k)}(\bar{x}) = F^{(k)}(\bar{x}) .$$

Therefore, (25) can be used to obtain the sequence of vectors $F^{(n)}(\bar{x})$ of (16a) recursively in terms of the given sequence of vectors $F_n(\bar{x}) = [F_n(x)]_{x=\bar{x}}$ given by (15a). This can be best visualized from the triangle of Fig. 1. For example,

$$F^{(1)} = F_1 + L_1 F_0 \quad (27a)$$

$$F_1^{(1)} = F_2 + L_1 F_1 + L_2 F_0 \quad (27b)$$

$$F^{(2)} = F_1^{(1)} + L_1 F^{(1)} \quad (27c)$$

$$F_2^{(1)} = F_3 + L_1 F_2 + 2L_2 F_1 + L_3 F_0 \quad (27d)$$

$$F_1^{(2)} = F_2^{(1)} + L_1 F_1^{(1)} + L_2 F^{(1)} \quad (27e)$$

$$F^{(3)} = F_1^{(2)} + L_1 F^{(2)} \quad (27f)$$

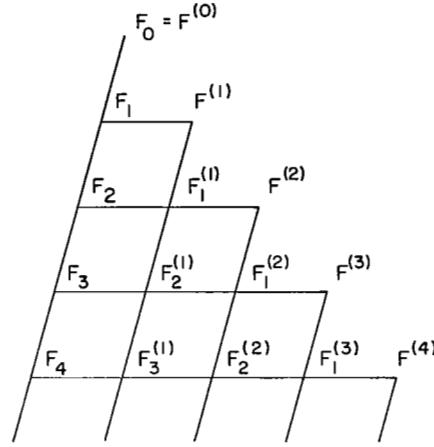


Fig. 1. RECURSIVE TRANSFORMATION
OF AN ANALYTIC FUNCTION UNDER EQ.
(14).

Now, using (16a), (16b) with $F = x$, and (14), followed by differentiation with respect to t , we obtain the following relations for x , \bar{x} , g , and f of (12) and (13)

$$x = \bar{x} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \bar{x}^{(n)}(\bar{x}) \quad (28a)$$

$$\dot{x} = \dot{\bar{x}} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} q^{(n)}(\bar{x}) \quad (28b)$$

and

$$g_0(\bar{x}) = f_0(\bar{x}) \quad (29a)$$

$$g_n(\bar{x}) = f^{(n)}(\bar{x}) - q^{(n)}(\bar{x}) , \quad n \geq 1 , \quad (29b)$$

where

$$\bar{x}^{(n)}(\bar{x}) = \left[\frac{d^{n-1}}{d\eta^{n-1}} W \right]_{\eta=0} \quad (30)$$

and

$$q^{(n)}(\bar{x}) = \sum_{m=1}^n C_m^n \bar{x}^{(m)} \cdot g_{n-m} . \quad (31)$$

Finally, the inverse transformation can be written* as

$$\bar{x} = x + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} x^{(n)}(x) . \quad (32)$$

To find the relation between the $x^{(n)}$'s and $\bar{x}^{(n)}$'s, one may eliminate $x - \bar{x}$ between (28a) and (32) and define the vector $u(x; \epsilon)$ as follows:

$$u(x; \epsilon) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} x^{(n)}(x) \quad (33a)$$

$$= - \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \bar{x}^{(n)}(\bar{x}) . \quad (33b)$$

Comparison of the above equation with (15) and (16) leads to

*

This is useful in constructing the integrals of motion, the initial conditions of (13) from the corresponding initial conditions of (12), etc.

$$u_0 = u^{(0)} = 0 \quad (34a)$$

$$u_n(x) = x^{(n)}(x) , \quad n \geq 1 \quad (34b)$$

$$u^{(n)}(\bar{x}) = - \bar{x}^{(n)}(\bar{x}) , \quad n \geq 1 . \quad (34c)$$

Chapter III

SIMPLIFIED GENERAL EXPANSIONS

Given the sequence of vectors F_n , F_{n-1} , ..., and F_0 , there are two separate approaches to the construction of the required sequence of vectors $F^{(n)}$ ($n \geq 0$). The first is Deprit's approach [1] for Hamiltonian equations; Deprit introduces the auxiliary functions $F_n^{(k)}$ and moves recursively from the left diagonal of Fig. 1 to the right diagonal. In the second approach, introduced by the author [2], one constructs $F^{(n)}$ ($n \geq 0$) only in terms of F_n , $F^{(n-1)}$, ..., and $F^{(0)}$ by introducing a suitable linear operator. This approach was found useful in constructing the inverse transformation and simplified general expansions. To show how this can be done, let us write (25) as

$$F_n^{(k)} = F_{n-1}^{(k+1)} - \sum_{m=0}^{n-1} C_m^{n-1} L_{m+1} F_{n-m-1}^{(k)} ; \quad n \geq 1, \quad k \geq 0 . \quad (35)$$

By successive elimination of the vectors on the right-hand side of the above equation, one would eventually obtain $F_n^{(k)}$ in terms of $F^{(k+n)}$, $F^{(k+n-1)}$, ..., and $F^{(k)}$. Thus one may assume for $F_n^{(k)}$ the form

$$F_n^{(k)} = F^{(k+n)} - \sum_{j=1}^n C_j^n G_j F^{(k+n-j)} ; \quad n \geq 1, \quad k \geq 0 , \quad (36)$$

where G_j is a linear operator which is a function of L_j, L_{j-1}, \dots , and L_1 . Substitution of (36) into (35) yields for G_j the recursion relation

$$G_j = L_j - \sum_{0 \leq m \leq j-2} C_m^{j-1} L_{m+1} G_{j-m-1} , \quad 1 \leq j \leq n . \quad (37)$$

For example,

$$G_1 = L_1 \quad (38a)$$

$$G_2 = L_2 - L_1 L_1 \quad (38b)$$

$$G_3 = L_3 - L_1 (L_2 - L_1 L_1) - 2L_2 L_1 \quad (38c)$$

For $k = 0$ and $k = 1$, (36) yields

$$F^{(n)} = F_n + \sum_{j=1}^n C_j^n G_j F^{(n-j)}; \quad n \geq 1 \quad (39)$$

$$F_n^{(1)} = F^{(n+1)} - \sum_{j=1}^n C_j^n G_j F^{(n-j+1)}, \quad n \geq 1 \quad (40)$$

Also, if we define $G_j F^{(i)}$ as $F_{j,i}$, (39) and (40) can be written in the form

$$F^{(n)} = F_n + \sum_{j=1}^n C_j^n F_{j,n-j} \quad (41)$$

$$F_n^{(1)} = F^{(n+1)} - \sum_{j=1}^n C_j^n F_{j,n-j+1}, \quad (42)$$

where

$$F_{j,i} = L_j F^{(i)} - \sum_{0 \leq m \leq j-2} C_m^{j-1} L_{m+1} F_{j-m-1,i} \quad (43)$$

Combining Eqs. (14) and (20) and making use of (21), (22a), and (42) with $F = \bar{x}$, we are lead to the general recursive relation for $\bar{x}^{(n)}$ of (28a)

$$\bar{x}^{(n)} = W_n + \sum_{1 \leq j \leq n-1} C_j^{n-1} \bar{x}_{j,n-j}, \quad n \geq 1 \quad (44a)$$

where

$$\bar{x}_{j,i} = L_j \bar{x}^{(i)} - \sum_{0 \leq m \leq j-2} C_m^{j-1} L_{m+1} \bar{x}_{j-m-1,i} . \quad (44b)$$

Using (41) with $F = u$ of (33), we obtain for $x^{(n)}$ the general formula

$$x^{(n)} = -\bar{x}^{(n)} + \sum_{1 \leq j \leq n-1} C_j^n \bar{x}_{j,n-j}; \quad n \geq 1 , \quad (45)$$

where $\bar{x}_{j,n-j}$ is as defined for (44b). Now, $x^{(n)}(x)$ of (32) is simply given by

$$x^{(n)}(x) = [x^n]_{\bar{x}=x} . \quad (46)$$

Using (29), (31), and (42), one should obtain (after some relatively tedious algebraic manipulation) the following simplified general recursion relation between the vector g of (13) and the vector f of (12).

$$g_0(\bar{x}) = f_0(\bar{x}) \quad (47a)$$

$$g_n(\bar{x}) = f_n(\bar{x}) + \sum_{j=1}^{n-1} \left[C_{j-1}^{n-1} L'_j f_{n-j} + C_j^{n-1} g_{j,n-j} \right] - L'_{f_0} w_n , \quad (47b)$$

where

$$g_{j,i} = L'_j g_i - \sum_{m=0}^{j-2} C_m^{j-1} L'_{m+1} g_{j-m-1,i} , \quad (48)$$

$$L'_j F = F_{\bar{x}} \cdot W_j - W_{j\bar{x}} \cdot F , \quad (49a)$$

and

$$L'_f W_n = -L'_n f_0 \quad (49b)$$

Note that Eq. (41) does not require the F_n 's to be the given vectors. In fact, (41) has the nice property of constructing the F_n 's from the $F^{(n)}$'s. This can be valuable in reducing the computation requirements when the given perturbation is limited in order. To show how this can be achieved, let $\dot{\bar{x}} = g(\bar{x}; \epsilon) = g_0 + \epsilon g_1$ be the given differential equations, then in view of (48), we find $g_{j,i} = 0$ for $i \geq 2$ and $j \geq 1$. Making use of this fact and Eq. (47), we acquire the desired reduced formula for constructing the transformed differential equations $\dot{x} = f(x; \epsilon)$, where $f(x; \epsilon) = f(\bar{x}; \epsilon)|_{\bar{x}=x}$.

Equations (47) to (49) are directly applicable to nonlinear resonant problems in which*

$$x = \begin{bmatrix} \alpha \\ \theta \end{bmatrix} \quad (50a)$$

$$\dot{x} = f = \begin{bmatrix} \epsilon u(\alpha, \theta; \epsilon) \\ \omega(\alpha) + \epsilon v(\alpha, \theta; \epsilon) \end{bmatrix}, \quad (50b)$$

where $u(\alpha, \theta; \epsilon)$ and $v(\alpha, \theta; \epsilon)$ are periodic in θ . It is desirable to transform to a new vector

$$\bar{x} = \begin{bmatrix} \bar{\alpha} \\ \bar{\theta} \end{bmatrix}, \quad (51)$$

*

A second-order expansion for this system of differential equations was obtained by Morrison [9], using a technique similar to that developed by Krylov and Bogoliubov [5].

so that the resulting $g_n (n \geq 1)$ will contain only certain slowly varying combinations of the $\bar{\theta}$ elements. Equations (47) to (49) can be used to define the W_n 's successively so as to remove all "short-period" terms from the g_n 's; such a W_n is unique up to an arbitrary additive long-period vector. It should be mentioned that, in the present case, the W_n 's are easily obtainable through solutions of simple linear partial-differential equations of first order.

When no resonances occur, one can construct the transformation

$$x = x(\bar{x}; \epsilon) , \quad (52a)$$

which reduces (50b) to the form

$$\dot{\bar{x}} = g = \begin{bmatrix} \epsilon u(\bar{\alpha}, -; \epsilon) \\ \omega(\bar{\alpha}) + \epsilon v(\bar{\alpha}, -; \epsilon) \end{bmatrix} . \quad (52b)$$

Then the solution of the original system (50b) reduces to solving the differential system of equations $\dot{\bar{\alpha}} = \epsilon u(\bar{\alpha}; \epsilon)$ and quadratures for $\bar{\theta}$. Through the transformation (52a), we get the solution of (50b). In this case, the initial conditions for (52b) are obtainable from the corresponding initial conditions of x through the inverse transformation (32).

It is interesting to observe that, if we replace x by $\begin{bmatrix} x \\ X \\ t \end{bmatrix}$, \bar{x} by $\begin{bmatrix} y \\ Y \\ t \end{bmatrix}$, and W by $\begin{bmatrix} W_X \\ -W_Y \\ 0^x \end{bmatrix}$ and if f can be generated from a Hamiltonian

$$H(x, X, t; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(x, X, t), \quad \text{i.e.,} \quad f = \begin{bmatrix} H_X \\ -H_Y \\ 1^x \end{bmatrix} ,$$

then g can also be generated from a Hamiltonian.

$$K(y, Y, t; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} K_n(y, Y, t), \quad \text{i.e.,} \quad g = \begin{bmatrix} K_Y \\ -K_Y \\ 1^y \end{bmatrix} ,$$

such that (47) to (49) reduce to the scalar form [2]

$$K_0 = H_0(y, Y, t) \quad (53a)$$

$$K_n = H_n(y, Y, t) + \sum_{j=1}^{n-1} \left[C_{j-1}^{n-1} L_j'' H_{n-j} + C_j^{n-1} K_{j, n-j} \right] - \frac{DW_n}{Dt}, \quad (53b)$$

where

$$L_j'' f = f_y \cdot W_Y - f_Y \cdot W_y, \quad (53c)$$

$$\frac{DW_n}{Dt} = W_{nt} - L_n'' H_0, \quad (53d)$$

and

$$K_{j,i} = L_j'' K_i - \sum_{m=0}^{j-2} C_m^{j-1} L_{m+1}'' K_{j-m-1, i}. \quad (53e)$$

Thus, one may say that the theory of perturbation based on Lie transforms is an important special case of the present theory in which the analysis of a $2N$ -element vector is reduced to the analysis of a single scalar function, the Hamiltonian.

The foregoing analysis now suggests a general simplified procedure for systems which are mainly represented by a Hamiltonian with some lesser perturbing forces nonderivable from a potential. To show how this can be done, let us take the system described by (8a) and (8b), obtained for the physical system described by (1). The procedure suggested here is to tackle the problem in two steps:

- (1) A canonical transformation from $(x, X) \rightarrow (y, Y)$ in which we can apply (53) and define the W_n 's successively so as to remove all the undesirable terms (e.g., short-period terms) from the K_n 's. Since the transformation is canonical, we should satisfy the constraint

$$\dot{\mathbf{x}} \cdot \delta \mathbf{X} - \dot{\mathbf{X}} \cdot \delta \mathbf{x} - \delta H = \dot{\mathbf{y}} \cdot \delta \mathbf{Y} - \dot{\mathbf{Y}} \cdot \delta \mathbf{y} - \delta K \quad (54)$$

To allow for the vectors $-\epsilon \mathbf{Q} \cdot \mathbf{q}_{\mathbf{X}}$ and $\epsilon \mathbf{Q} \cdot \mathbf{q}_{\mathbf{x}}$ on the right-hand side of (8), we should add the scalar quantity $\epsilon \mathbf{Q} \cdot \delta \mathbf{q}$ to both sides of (54). If

$$\mathbf{Q} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathbf{Q}_n(\mathbf{x}, \mathbf{X}, t) \quad ,$$

then, by using (16a) with $\mathbf{F} = \mathbf{Q}$ and \mathbf{q} , we get

$$\dot{\mathbf{y}} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathbf{v}_n(\mathbf{y}, \mathbf{Y}, t) \quad (55a)$$

$$\dot{\mathbf{Y}} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathbf{V}_n(\mathbf{y}, \mathbf{Y}, t) \quad , \quad (55b)$$

where

$$\mathbf{v}_n = \mathbf{K}_{n\mathbf{Y}} - \sum_{m=0}^{n-1} n \mathbf{C}_m^{n-1} \mathbf{Q}^{(m)} \cdot \mathbf{q}_{\mathbf{Y}}^{(n-m-1)} \quad , \quad (56a)$$

$$\mathbf{V}_n = -\mathbf{K}_{n\mathbf{y}} + \sum_{m=0}^{n-1} n \mathbf{C}_m^{n-1} \mathbf{Q}^{(m)} \cdot \mathbf{q}_{\mathbf{y}}^{(n-m-1)} \quad ; \quad (56b)$$

$$\mathbf{q}^{(n)} = \sum_{j=1}^n \mathbf{C}_j^n \mathbf{q}_{j, n-j} \quad , \quad n \geq 1 \quad (57a)$$

$$\mathbf{q}_{j, i} = \mathbf{L}_j'' \mathbf{q}^{(i)} - \sum_{m=0}^{j-2} \mathbf{C}_m^{j-1} \mathbf{L}_{m+1}'' \mathbf{q}_{j-m-1, i} \quad , \quad (57b)$$

$$q^{(0)} = q(y, Y) \quad ; \quad (57c)$$

$$Q^{(n)} = Q_n(y, Y, t) + \sum_{j=1}^n C_j^n Q_{j, n-j}, \quad n \geq 0 \quad (58a)$$

and

$$Q_{j,i} = L_j'' Q^{(i)} - \sum_{m=0}^{j-2} C_m^{j-1} L_{m+1}'' Q_{j-m-1, i} \quad (58b)$$

(2) A noncanonical transformation from (y, Y) to z in which

$$\dot{z} = g(z, t; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} g_n(z, t). \quad (59)$$

Now, from Eqs. (47) to (49), we get

$$g_0(z, t) = f_0(z, t)$$

$$g_n(z, t) = f_n(z, t) + \sum_{j=1}^n \left[C_{j-1}^{n-1} L_j' f_{n-j} + C_j^{n-1} g_{j, n-j} \right] - \frac{DS_n}{Dt}, \quad (60a)$$

where

$$f_n(z, t) = \begin{bmatrix} v_n \\ V_n \end{bmatrix} \quad n \geq 0, \quad (60b)$$

$$g_{j,i} = L_j' g_i - \sum_{m=0}^{j-2} C_m^{j-1} L_{m+1}' g_{j-m-1, i}, \quad (60c)$$

$$\frac{DS_n}{Dt} = S_{nt} - L_n' f_0, \quad (60d)$$

and the rest of the undesirable terms in the f_n 's can then be eliminated by the generating vectors S_n 's.

It should be noted that, by using (41), one can express any vector

$$u(x, X, t; \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} u_n(x, X, t)$$

directly in terms of z, t and ϵ , since we have

$$\begin{aligned} u(x, X, t; \epsilon) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} u_n(x, X, t) \\ &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} u^{(n)}(y, Y, t) \\ &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} U^{(n)}(z, t) \quad , \end{aligned} \quad (61)$$

where

$$u^{(n)}(y, Y, t) = u_n(y, Y, t) + \sum_{j=1}^n C_j^n u_{j, n-j} \quad , \quad (62a)$$

$$U^{(n)}(z, t) = [u^{(n)}(y, Y, t)]_{\substack{y \\ Y}=z} + \sum_{j=1}^n C_j^n U_{j, n-j} \quad , \quad (62b)$$

$$u_{j, i} = L_j^n u^{(i)} - \sum_{m=0}^{j-2} C_m^{j-1} L_{m+1}^n u_{j-m-1, i} \quad , \quad (63a)$$

$$U_{j, i} = L_j U^{(i)} - \sum_{m=0}^{j-2} C_m^{j-1} L_{m+1} U_{j-m-1, i} \quad (63b)$$

Chapter IV

PROCEDURE OUTLINE

Consider the system of differential equations in the standard form

$$\dot{x} = f_0(x) + \epsilon f_1(x) + \frac{1}{2!} \epsilon^2 f_2(x) + \frac{1}{3!} \epsilon^3 f_3(x) + \dots \quad (64)$$

The essence of the technique proposed here is to construct the mapping $(x; \epsilon) \rightarrow (\bar{x}; \epsilon)$, analytic in ϵ at $\epsilon = 0$, so as to achieve specific requirements (e.g., elimination of short-period terms, suppression of all angle coordinates, and so on) in the transformed system of differential equations

$$\dot{\bar{x}} = g_0(\bar{x}) + \epsilon g_1(\bar{x}) + \frac{1}{2!} \epsilon^2 g_2(\bar{x}) + \frac{1}{3!} \epsilon^3 g_3(\bar{x}) + \dots \quad (65)$$

We plan to build the mapping explicitly in the form of power series

$$x = \bar{x} + \epsilon \bar{x}^{(1)}(\bar{x}) + \frac{1}{2!} \epsilon^2 \bar{x}^{(2)}(\bar{x}) + \frac{1}{3!} \epsilon^3 \bar{x}^{(3)}(\bar{x}) + \dots, \quad (66)$$

together with its inverse

$$\bar{x} = x + \epsilon x^{(1)}(x) + \frac{1}{2!} \epsilon^2 x^{(2)}(x) + \frac{1}{3!} \epsilon^3 x^{(3)}(x) + \dots \quad (67)$$

Under such mapping, any analytic vector $F(x; \epsilon)$ given by

$$F(x; \epsilon) = F_0(x) + \epsilon F_1(x) + \frac{1}{2!} \epsilon^2 F_2(x) + \frac{1}{3!} \epsilon^3 F_3(x) + \dots \quad (68)$$

can be built in the form

$$F(x; \epsilon) = F^{(0)}(\bar{x}) + \epsilon F^{(1)}(\bar{x}) + \frac{1}{2!} \epsilon^2 F^{(2)}(\bar{x}) + \frac{1}{3!} \epsilon^3 F^{(3)}(\bar{x}) + \dots \quad (69)$$

We shall describe in full detail the operations performed to carry the transformation up to third order in ϵ . The scheme is basically a recursive one and it is initiated by putting

$$\begin{aligned} g_0(\bar{x}) &= f_0(\bar{x}) \\ F^{(0)}(\bar{x}) &= F_0(\bar{x}) \end{aligned} \quad (70)$$

The first-order operation begins by considering the linear partial-differential relation

$$g_1(\bar{x}) = f_1(\bar{x}) - L'_{f_0} W_1 \quad (71)$$

Assuming that a choice has been made for $g_1(\bar{x})$, we then solve for $W_1(\bar{x})$ and compute

$$\begin{aligned} \bar{x}^{(1)} &= W_1 \\ x^{(1)} &= -\bar{x}^{(1)} \\ F_{1,0} &= L_1 F^{(0)} \\ F^{(1)} &= F_1 + F_{1,0} \end{aligned} \quad (72)$$

To prepare for the second-order expansion, we compute

$$g_{1,1} = L'_1 g_1$$

At the second-order level, we set up the partial-differential relation

$$g_2 = f_2 + L'_1 f_1 + g_{1,1} - L'_{f_0} W_2 \quad (73)$$

The unknown vector g_2 is selected in compliance with the goals proposed for the transformation, and the resulting linear partial-differential equation is integrated to yield $W_2(\bar{x})$. The step of second order is completed by computing

$$\begin{aligned}
 \bar{x}_{1,1} &= L_1 \bar{x}^{(1)} \\
 \bar{x}^{(2)} &= W_2 + \bar{x}_{1,1} \\
 x^{(2)} &= -\bar{x}^{(2)} + 2\bar{x}_{1,1} \\
 F_{1,1} &= L_1 F^{(1)} \\
 F_{2,0} &= L_2 F^{(0)} - L_1 F_{1,0} \\
 F^{(2)} &= F_2 + 2F_{1,1} + F_{2,0} \quad . \quad (74)
 \end{aligned}$$

To prepare for the third-order expansion, we compute

$$\begin{aligned}
 g_{1,2} &= L_1' g_2 \\
 g_{2,1} &= L_2' g_1 - L_1' g_{1,1} \quad . \quad (75)
 \end{aligned}$$

At third-order level, we form the partial-differential equation

$$g_3 = f_3 + L_1' f_2 + 2L_2' f_1 + 2g_{1,2} + g_{2,1} - L_{f0}' W_3 \quad . \quad (76)$$

The unknown vector is chosen, and the resulting partial-differential equation is solved to yield $W_3(\bar{x})$. Then the following sequence of operations will complete the third-order analysis.

$$\begin{aligned}
\bar{x}_{1,2} &= L_1 \bar{x}^{(2)} \\
\bar{x}_{2,1} &= L_2 \bar{x}^{(1)} - L_1 \bar{x}_{1,1} \\
\bar{x}^{(3)} &= W_3 + 2\bar{x}_{1,2} + \bar{x}_{2,1} \\
x^{(3)} &= -\bar{x}^{(3)} + 3 \bar{x}_{1,2} + 3 \bar{x}_{2,1} ; \tag{77}
\end{aligned}$$

$$\begin{aligned}
F_{1,2} &= L_1 F^{(2)} \\
F_{2,1} &= L_2 F_1 - L_1 F_{1,1} \\
F_{3,0} &= L_3 F^{(0)} - L_1 F_{2,0} - 2L_2 F_{1,0} \\
F^{(3)} &= F_3 + 3F_{1,2} + 3F_{2,1} + F_{3,0} . \tag{78}
\end{aligned}$$

Clearly, the entire procedure can be extended to any order by using (12), (13), (15), (16), (28a), (32), (41), and (43) to (49). Note that f_0 is not arbitrary. The present theory is restricted to problems which can be reduced (in some way or another) to a standard form such as (12) in which f_0 allows a solution to the partial-differential equations defining the W_n . Some of these problems can be found in the literature [3,4,10,11].

Chapter V

EXAMPLES

In the first example,* we consider Van der Pol's equation

$$\ddot{q} + q = \epsilon(1 - q^2) \dot{q} . \quad (79)$$

According to the method of variation of parameters, we solve as a first step (79) for $\epsilon = 0$. This solution may be put in the form of

$$q = A \sin \varnothing \quad (80a)$$

$$\dot{q} = A \cos \varnothing . \quad (80b)$$

In the second step, we consider (80) as the solution of (79). Then, if we differentiate (80a) with respect to t and equate the resulting equation to (80b), we find the variation of the parameters A and \varnothing are governed by

$$\dot{A}S + A\dot{\varnothing}C = AC , \quad (81)$$

where

$$S = \sin \varnothing \quad \text{and} \quad C = \cos \varnothing .$$

Substitution of (80) into (79) also yields

$$\dot{A}C - A\dot{\varnothing}S = -AS + \epsilon AC[1 - A^2S^2] . \quad (82)$$

Equations (81) and (82) lead to the standard form

*The algebraic analysis in this example was carried out on the IBM 360 computer using REDUCE language [7].

$$\dot{A} = \epsilon \left[\frac{A}{2} \left(1 - \frac{A^2}{4} \right) + \frac{A}{2} C_2 + \frac{A^3}{8} C_4 \right] \quad (83a)$$

$$\dot{\phi} = 1 - \epsilon \left[\frac{1}{2} \left(1 - \frac{A^2}{2} \right) S_2 + \frac{1}{8} A^2 S_4 \right] , \quad (83b)$$

where

$$S_n = \sin n\phi \quad \text{and} \quad C_n = \cos n\phi .$$

Define $x = \begin{bmatrix} A \\ \phi \end{bmatrix}$, then (12) and (83) yield

$$f_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (84a)$$

$$f_1 = \begin{bmatrix} \frac{A}{2} \left(1 - \frac{A^2}{4} \right) + \frac{A}{2} C_2 + \frac{A^3}{8} C_4 \\ -\frac{1}{2} \left(1 - \frac{A^2}{2} \right) S_2 - \frac{1}{8} A^2 S_4 \end{bmatrix} \quad (84b)$$

$$f_n = 0 \quad n > 1 . \quad (84c)$$

Now, in the third step of the method of variation of parameters, it is desirable to transform from x to a new vector $\bar{x} = \begin{bmatrix} A \\ \bar{\phi} \end{bmatrix}$, so that

$$\dot{\bar{x}} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} g_n(\bar{x}) , \quad (85)$$

where g_n contains only secular terms. Up to second order, (47) to (49) yield

$$g_0 = f_0 \quad (86a)$$

$$g_1 = f_1 - w_1 \bar{\phi} \quad (86b)$$

$$g_2 = L_1'(f_1 + g_1) - W_2 \bar{\vartheta} , \quad (86c)$$

where

$$L_1' F = F_x^- \cdot W_1 - W_{1x}^- \cdot F . \quad (86d)$$

Choosing W_1 to eliminate the short-period part of g_1 , we obtain

$$W_1 = \begin{bmatrix} \frac{\bar{A}}{4} \bar{s}_2 + \frac{\bar{A}^3}{32} \bar{s}_4 \\ \frac{1}{4} \left(1 - \frac{\bar{A}^2}{2} \right) \bar{c}_2 + \frac{\bar{A}^2}{32} \bar{c}_4 \end{bmatrix} , \quad (87)$$

where

$$\bar{s}_n = \sin n\bar{\vartheta} \quad \text{and} \quad \bar{c}_n = \cos n\bar{\vartheta} .$$

Hence,

$$g_1 = \begin{bmatrix} \frac{\bar{A}}{2} \left(1 - \frac{\bar{A}^2}{4} \right) \\ 0 \end{bmatrix} . \quad (88)$$

Computing $L_1'(f_1 + g_1)$ and choosing W_2 to eliminate the short-period part of g_2 , we obtain

$$W_2 = \begin{bmatrix} -\frac{\bar{A}^5}{384} \bar{c}_6 + \frac{\bar{A}^3}{64} \bar{c}_4 + \frac{\bar{A}^3}{64} \left(7 - \frac{3\bar{A}^2}{2} \right) \bar{c}_2 \\ \frac{\bar{A}^4}{384} \bar{s}_6 - \frac{\bar{A}^2}{64} \left(1 - \frac{\bar{A}^2}{4} \right) \bar{s}_4 + \frac{\bar{A}^2}{32} \left(1 + \frac{\bar{A}^2}{2} \right) \bar{s}_2 \end{bmatrix} . \quad (89)$$

Hence,

$$g_2 = \begin{bmatrix} 0 \\ -\frac{11}{128} \bar{A}^4 + \frac{3}{8} \bar{A}^2 - \frac{1}{4} \end{bmatrix} . \quad (90)$$

Equations (85), (88), and (90) lead to

$$\frac{\dot{x}}{x} = \begin{bmatrix} \epsilon \frac{\bar{A}}{2} \left(1 - \frac{\bar{A}^2}{4} \right) \\ 1 + \epsilon^2 \left(-\frac{1}{8} + \frac{3\bar{A}^2}{16} - \frac{11\bar{A}^4}{256} \right) \end{bmatrix} . \quad (91)$$

To express q of (80a) in terms of \bar{A} and $\bar{\phi}$, one may use (15a), (16a), and (41). This leads to

$$q = q^{(0)} + \epsilon q^{(1)} + \frac{\epsilon^2}{2} q^{(2)} , \quad (92a)$$

where

$$q^{(0)} = \bar{A} \bar{S}_1 \quad (92b)$$

$$\begin{aligned} q^{(1)} &= L_1 q^{(0)} \\ &= \frac{\bar{A}}{4} \left(1 - \frac{\bar{A}^2}{4} \right) \bar{C}_1 - \frac{\bar{A}^3}{32} \bar{C}_3 \end{aligned} \quad (92c)$$

$$\begin{aligned} q^{(2)} &= L_2 q^{(0)} + L_1 q^{(1)} \\ &= \frac{\bar{A}}{16} \left[1 - \bar{A}^2 + \frac{15}{64} \bar{A}^4 \right] \bar{S}_1 \\ &\quad + \frac{\bar{A}^3}{16} \left[1 - \frac{5\bar{A}^2}{32} \right] \bar{S}_3 - \frac{5\bar{A}^5}{1536} \bar{S}_5 . \end{aligned} \quad (92d)$$

In the fourth step of the method of variation of parameters, we wish to solve the simple equations of (91). As an example, we consider the periodic solution obtained when

$$\bar{A} = 2 \quad (93a)$$

$$\bar{\theta} = \left(1 - \frac{\epsilon^2}{16}\right)t + \psi(\epsilon) \quad , \quad (93b)$$

where $\psi(\epsilon)$ is a free function of the small parameter ϵ to be chosen to satisfy some condition on q . Equations (92) and (93) now lead to

$$q = 2\bar{S}_1 - \frac{\epsilon}{4}\bar{C}_3 + \frac{\epsilon^2}{32}\left(\frac{3}{2}\bar{S}_1 + 3\bar{S}_3 - \frac{5}{3}\bar{S}_5\right) \quad . \quad (94)$$

Now, if we choose $\psi = \psi_0 + \epsilon\psi_1 + (\epsilon^2/2)\psi_2$ to satisfy $\dot{q}|_{t=0} = 0$, one obtains $\psi_0 = \pi/2$, $\psi_1 = -3/8$, and $\psi_2 = 0$. The corresponding q then takes the form

$$\begin{aligned} q = & \left(2 - \frac{3\epsilon^2}{32}\right) \cos \theta + \frac{3\epsilon}{4} \sin \theta \\ & + \frac{3\epsilon^2}{16} \cos 3\theta - \frac{\epsilon}{4} \sin 3\theta \\ & - \frac{5\epsilon^2}{96} \cos 5\theta \quad , \end{aligned} \quad (95a)$$

where

$$\theta = \left(1 - \frac{\epsilon^2}{16}\right)t \quad . \quad (95b)$$

Comparing the above equation with [12], one finds that the coefficient of $\cos \theta$ is $[2 - (\epsilon^2/8)]$ rather than $[2 - (3/32)\epsilon^2]$. The difference is due to some feedback from third-order analysis since the periodicity condition will yield $[\bar{A} = 2 - (\epsilon^2/32)]$ rather than 2.

It might be noted that the foregoing perturbation developments can be carried out directly in terms of the physical variables, i.e., q and $\dot{q} \equiv p$. To show how this can be achieved, we consider the nonlinear differential equation

$$\ddot{q} + q = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} h_n(q, \dot{q}) \quad . \quad (96)$$

Let

$$x = \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} A \sin \phi \\ A \cos \phi \end{bmatrix} \quad , \quad (97a)$$

and

$$\bar{x} = \begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} \bar{A} \sin \bar{\phi} \\ \bar{A} \cos \bar{\phi} \end{bmatrix} \quad . \quad (97b)$$

In view of (96), and by making use of the definitions of p and x , we obtain

$$\dot{x} = \begin{bmatrix} p \\ -q + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} h_n(q, p) \end{bmatrix} \quad . \quad (98)$$

Comparing (12) with (98), we get

$$f_0 = \begin{bmatrix} p \\ -q \end{bmatrix} \quad , \quad (99a)$$

$$f_n = \begin{bmatrix} 0 \\ h_n(q, p) \end{bmatrix} \quad n \geq 1 \quad . \quad (99b)$$

Let

$$W_n = \begin{bmatrix} W_{n_1} \\ W_{n_2} \end{bmatrix}. \quad (100)$$

Now, by using (100) and realizing the fact that $\bar{q} = -\partial \bar{p} / \partial \bar{\theta}$ and $\bar{p} = \partial \bar{q} / \partial \bar{\theta}$, Eq. (47b) can be reduced to the form

$$\begin{bmatrix} \frac{\partial W_{n_1}}{\partial \bar{\theta}} - W_{n_2} \\ \frac{\partial W_{n_2}}{\partial \bar{\theta}} + W_{n_1} \end{bmatrix} = \begin{bmatrix} F_{n_1}(\bar{q}, \bar{p}) \\ F_{n_2}(\bar{q}, \bar{p}) \end{bmatrix}, \quad (101a)$$

where

$$\begin{bmatrix} F_{n_1}(\bar{q}, \bar{p}) \\ F_{n_2}(\bar{q}, \bar{p}) \end{bmatrix} = f_n - g_n + \sum_{j=1}^{n-1} \left[C_{j-1}^{n-1} L_j' f_{n-j} + C_j^{n-1} g_{j, n-j} \right]. \quad (101b)$$

Equation (101a) now leads to

$$\frac{\partial^2 W_{n_1}}{\partial \bar{\theta}^2} + W_{n_1} = \frac{\partial F_{n_1}}{\partial \bar{\theta}} + F_{n_2}, \quad (102a)$$

$$W_{n_2} = \frac{\partial W_{n_1}}{\partial \bar{\theta}} - F_{n_1}. \quad (102b)$$

In view of (12) and (97b), g_n must be constructed in the form

$$\xi_n = \begin{bmatrix} a_n(\bar{A}) \sin \bar{\vartheta} + b_n(\bar{A}) \cos \bar{\vartheta} \\ a_n(\bar{A}) \cos \bar{\vartheta} - b_n(\bar{A}) \sin \bar{\vartheta} \end{bmatrix}, \quad (103)$$

where $a_n(\bar{A})$ and $b_n(\bar{A})$ are chosen to eliminate the coefficients of $\sin \bar{\vartheta}$ and $\cos \bar{\vartheta}$ from $(\partial F_{n1}/\partial \bar{\vartheta}) + F_{n2}$ of (102a) (otherwise W_{n1} will be unbounded). Note that the operations involved in obtaining F_{n1} and F_{n2} of (101b) are carried out in terms of \bar{q} and \bar{p} . Then, for the computation of W_{n1} and W_{n2} , we substitute \bar{q} and \bar{p} in terms of \bar{A} and $\bar{\vartheta}$. After these computations have been performed, we go back from \bar{A} and $\bar{\vartheta}$ to \bar{q} and \bar{p} to obtain $W_{n1}(\bar{q}, \bar{p})$ and $W_{n2}(\bar{q}, \bar{p})$ which are to be used in the operations of the next step. The cycle is then repeated until we reach the desired order of perturbation. Now, substitution of \bar{q} and \bar{p} in terms of \bar{A} and $\bar{\vartheta}$ in (12) will lead to the desired equations for $\dot{\bar{A}}$ and $\dot{\bar{\vartheta}}$. Also, the vector \bar{x} can be constructed in terms of \bar{x} by using (32) and (44).

The transformation of the ordinary differential equation (96) to the partial-differential equations (102a) is similar, in spirit, to the methods of Poincare' [13], Kevorkian [14], and Nayfeh [15]. The equivalency, however, is still an open question.

As an application of the above procedure, we consider the linear differential equation

$$\ddot{q} + q = -2\epsilon \dot{q}, \quad (104)$$

whose exact solution is given by

$$q = \bar{A}_0 \bar{e}^{\epsilon t} \sin \left[(1 - \epsilon^2)^{1/2} t + \bar{\vartheta}_0 \right], \quad (105)$$

where \bar{A}_0 and $\bar{\vartheta}_0$ are arbitrary constants. Using Eqs. (96) to (104), one can easily verify that the second-order analysis leads to the following results.

$$\begin{aligned}
g_0 &= \begin{bmatrix} \bar{p} \\ -\bar{q} \end{bmatrix}, \\
g_1 &= \begin{bmatrix} -\bar{q} \\ -\bar{p} \end{bmatrix}, & w_1 &= \begin{bmatrix} 0 \\ -\bar{q} \end{bmatrix}, \\
g_2 &= \begin{bmatrix} -\bar{p} \\ \bar{q} \end{bmatrix}, & w_2 &= \begin{bmatrix} 0 \\ -\bar{p} \end{bmatrix}.
\end{aligned}$$

In view of (13), (97b), and (106), we get

$$\begin{bmatrix} \dot{\bar{q}} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} -\epsilon \bar{q} + (1 - \epsilon^2/2) \bar{p} \\ -\epsilon \bar{p} - (1 - \epsilon^2/2) \bar{q} \end{bmatrix}. \quad (107)$$

Substitution of \bar{q} and \bar{p} in terms of \bar{A} and $\bar{\vartheta}$ in (107) leads to

$$\begin{aligned}
\dot{\bar{A}} &= -\epsilon \bar{A}, \\
\dot{\bar{\vartheta}} &= 1 - \epsilon^2/2.
\end{aligned} \quad (108)$$

The solution to (108) implies that

$$\begin{aligned}
\bar{q} &= \bar{A}_0 e^{-\epsilon t} \sin \left[(1 - \epsilon^2/2)t + \bar{\vartheta}_0 \right] \\
\bar{p} &= \bar{A}_0 e^{-\epsilon t} \cos \left[(1 - \epsilon^2/2)t + \bar{\vartheta}_0 \right].
\end{aligned} \quad (109)$$

Using (28a) and (44) with $x = \begin{bmatrix} q \\ p \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} \bar{q} \\ \bar{p} \end{bmatrix}$, we obtain

$$q = \bar{q} \tag{110}$$

$$p = -\epsilon \bar{q} + (1 - \epsilon^2/2) \bar{p} .$$

This is seen to agree with the exact solution to within $O(\epsilon^3)$ in the frequency of oscillation.

Chapter VI

CONCLUSION

A perturbation expansion to arbitrary order is presented. The application to nonlinear oscillation problems has been discussed and illustrated by two examples. Although our investigation has been limited to systems described by ordinary differential equations, the method is applicable to some other systems described by partial-differential equations, as in the fields of plasma oscillations, fluid stability, etc.

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